Implementation of parametric analysis of OFC and RHS parameters of LP models to support operations management decisions using AIMMS

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Abstract

Although LP sensitivity analysis provides good inside on the effect of small changes in OFC or RHS parameters, in some cases the resulted ranges may be too tight for decision support, thus information about a wider range may be useful. For this purpose, a parametric analysis of the parameters must be considered. For doing this a good practical tool is provided by the AIMMS mathematical modelling system which is widely used for solving commercial optimization problems.

The objective of this paper is to show, how parametric analysis of the OFC and RHS parameters can be performed correctly using AIMMS.

Keywords: Decision support, LP Sensitivity analysis, AIMMS

Introduction

Organizations all over the world use business analytics (BA) to gain insight in order to drive business strategy and planning. The field of business analytics offers endless possibilities today and prescriptive analytics is the latest development in this field.

Allocation of scarce resources is a typical problem often encountered by managers and linear programming (LP) is a widely used tool for supporting the decision making in this matter. LP models have some limitations: the objective function and the constraints on variables must be formulated using linear expressions and the decision variables have to be continuous. Although in practical cases, the linearity of constraints and the continuity of the decision variables may not hold, LP models are considered as a good approximation. In the era of Big Data ever more data is collected and is available for use in decision support models. Larger data implies larger models and quick, often instant decision must be made which requires the model to be solved quickly and managers also must deal with the uncertainty of the input parameters.

The use of an LP model has two major advantages over the use of more complex Mixed Integer Linear Programing (MILP) model: computation time is less hardware and time consuming and further valuable insight can be gained about the problem using sensitivity analysis. The standard form of an LP problem as described by Hiller and Lieberman (1995) is the following:

$$\max(\mathbf{c}^T\mathbf{x}), \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0 \qquad (1)$$

where $\mathbf{x}$ is the decision variable vector, $\mathbf{c}$ is the objective function coefficients (OFC) vector, $\mathbf{b}$ is the right-hand-side (RHS) parameter vector. Table 1 contains all the notation used in the following part of the paper. Every linear programming problem, referred to as a primal problem, can be converted into a dual problem, which provides an upper bound to the optimal value of the primal problem. The standard form of the dual problem as described by Hiller and Lieberman (1995):

$$\min(\mathbf{b}^T\mathbf{y}), \mathbf{A}^T\mathbf{y} \geq \mathbf{c}, \mathbf{y} \geq 0 \qquad (2)$$

The optimal solution of an LP problem provides the optimal allocation of limited resources, while the optimal solution of the dual problem provides information about the marginal change of the objective function of the primal problem (shadow price), if a right-hand-side parameter changes.

Table 1 – Summary of notation

$\mathbf{A}$	Coefficient matrix
$\mathbf{b}$	Righ-hand-side vector with elements b_j $(j = 1,..,J)$
$\mathbf{c}$	Objective function coefficient vector with elements c_i $(i = 1,..,I)$
$\boldsymbol{x}$	Variable vector of the primal problem with elements x_i $(i = 1,..,I)$
$\mathbf{y}$	Variable vector of the dual problem with elements y_j $(j = 1,..,J)$
I^k	Intervals with constant rate of change of the objective value function $(k = 1,..,K)$
I^k_{rate}	Rate of change of the objective value function within the interval I^k
I^k_s	Value of the objective function at the starting point of interval I^k
I^k_e	Value of the objective function at the ending point of interval I^k
$LP(\lambda \leftarrow \nu)$	Modified version of the original LP, where parameter λ is modified to ν
$OF^*(\lambda \leftarrow \nu)$	Optimal value of the $LP(\lambda \leftarrow \nu)$ problem
$\xi_j^+(z)$	Maximal allowed increase of the z right-hand-side parameter of constraint j to remain within the Type III invariancy interval related to the modified $LP(b_j \leftarrow z)$ problem
$\xi_j^-(z)$	Maximal allowed decrease of the z right-hand-side parameter of constraint j to remain within the Type III invariancy interval related to the modified $LP(b_j \leftarrow z)$ problem
$\gamma_i^+(z)$	Maximal allowed increase for the z objective function coefficient of variable i to remain within the Type III invariancy interval related to the modified $LP(c_i \leftarrow z)$ problem
$\gamma_i^-(z)$	Maximal decrease allowed for the z objective function coefficient of variable i to remain within the Type III invariancy interval related to the modified $LP(c_i \leftarrow z)$ problem
$\mathbf{e}_j$	Unit vector with J elements, where $e_j = 1$ and $e_k = 0 \| k \neq j$
β_j^+	Maximal feasible increase of parameter b_j
β_j^-	Maximal feasible decrease of parameter b_j

Finding the optimal solution is just the first step. Since many of the parameters involved in the models are generally approximations, expectations or forecasts based on statistically available data, managers must deal also with the uncertainty in the available data. Sensitivity analysis provides information about the validity range of the primal and dual optimum. The validity range of the objective function coefficients (OFC) provides a range for each coefficient, within which the primal optimal solution will not change. Validity range of the right-hand-side(RHS) elements provides a range for each right-hand-side element. Within this range the dual optimum will not change.

The theoretical problems of sensitivity analysis under degeneracy are well known in the literature (Gal, 1986; Jansen et al., 1997). Many papers demonstrate erroneous management decisions based on the misinterpretation of sensitivity analysis results (Jansen et al., 1997). Koltai and Terlaky (2000) classified three types of sensitivity information. In non-degenerate cases the three types of sensitivities are identical, but in degenerate cases different sensitivity information could be provided by solvers. Most of the commercial LP solvers provide only type I sensitivity information but from a management standpoint type III sensitivity information are far more important.

Although sensitivity analysis provides good inside on the effect of small changes in OFC or RHS parameters, in some cases the resulted ranges may be too tight for decision support, thus information about a wider range may be useful. For this purpose, a parametric analysis of OFC and RHS parameters must be considered. The result of such an analysis is a set of consecutive intervals ($I^k, k = 1..K$) with constant rate of change of the objective value function within each interval. For each interval the rate of change of the objective function (I^k_{rate}) and the value of the objective function at the starting and ending points must be calculated (I^k_s and I^k_e).

The remainder of this paper is organized as follows. First, the building blocks required to perform a parametric analysis of LP models are discussed. Next, the algorithms for calculating the consecutive intervals of the objective value function for the RHS and OFC parameters are presented. Finally, the AIMMS implementation is described and an illustration example is provided.

Building blocks

Consider the $\max(\mathbf{c}^T\mathbf{x})$, $\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq 0$ linear program as the original LP to be solved, and $c_1, c_2, \dots, c_I$ are the elements of the **c** OFC vector, furthermore, $b_1, b_2, \dots, b_J$ are the elements of the b RHS parameters vector.

As an initial step, a parametric model for solving the original LP with modified RHS or OFC values must be implemented. Let $LP(\lambda \leftarrow \nu)$ note a modified version of the original LP where parameter y is modified to z and let $OF^*(\lambda \leftarrow \nu)$ be the corresponding optimal objective value of the modified LP.

For the calculation of the correct sensitivity ranges even in the degenerate case, a parametrized version of the additional LP's described be Koltai and Tatay (2011) must also be implemented.

Type III sensitivity provides information about the invariance of the rate of change of the objective value function. Let $\xi_j^+(z)$ and $\xi_j^-(z)$ denote the maximal increase and the maximal decrease allowed for the z right-hand-side parameter of constraint j to remain within the Type III invariancy interval related to the modified $LP(b_j \leftarrow z)$ problem.

Similarly let $\gamma_i^+(z)$ and $\gamma_i^-(z)$ denote the maximal increase and the maximal decrease allowed for the z objective function coefficient of variable i to remain within the Type III invariancy interval related to the modified $LP(c_i \leftarrow z)$ problem.

Algorithm for calculating consecutive RHS intervals

Intervals are calculated starting from the original right-hand-side parameter separately for increasing and decreasing directions and only when the original LP has at least one feasible solution.

Change on the right-hand-side parameter in one direction causes the feasible region of the LP to shrink, while changes on the opposite direction expand the feasible region while the corresponding constraint is active. After removing the corresponding constraint, the LP problem may become unbounded. In this case the length of the last interval will become infinite.

As an initial step, the maximal feasible change of the RHS parameter in the selected direction that allows the LP problem to remain feasible must be calculated.

Table 2 –calculating the maximal feasible RHS change

maximal feasible increase	maximal feasible decrease
$\mathbf{Ax} \leq \mathbf{b} - b_j\mathbf{e}_j + \beta\mathbf{e}_j$ $\beta \geq 0$ $\mathbf{x} \geq 0$ $\mathbf{max}(\beta)$	$\mathbf{Ax} \leq \mathbf{b} - b_j\mathbf{e}_j - \beta\mathbf{e}_j$ $\beta \geq 0$ $\mathbf{x} \geq 0$ $\mathbf{max}(\beta)$

To calculate these values an additional LP must be solved for each RHS parameter and each direction. For each constraint j, the maximal increases are determined by the LP problems of the first column of Table 2, while the maximal decreases are determined by the second column of Table 2. The maximal increase, as well as the maximal decrease are nonnegative numbers. The difference between these additional LP's and the original LP consists in using constraint j as an objective value function instead of being a constraint. Let β_j^+ and β_j^-notes the maximal feasible increase and the maximal feasible decrease respectively. One of these values may be infinite.

The pseudo-code for calculating subsequent intervals is presented in Table 3. The first column presents the algorithm for collecting increasing RHS intervals, while the second column presents the algorithm for collecting decreasing intervals. The rate of change denoted by SP^+ and SP^- are the right and left shadow prices of the modified $LP(b_j \leftarrow b_j^k)$ problem.

After the initialization step where the maximal feasible modification of the RHS parameter is calculated, the following steps are repeated until the calculated maximal feasible modification is reached, or the maximum increase/decrease is infinite:

- create and solve the modified $LP(b_j \leftarrow b_j^k)$ problem,
- calculate Type III range for the required direction,
- collect interval data.

Algorithm for calculating consecutive OFC intervals

The modification of an OFC parameter does not influence the feasibility of the LP problem, however, after a certain point, the previously bounded problem may become unbounded.

The pseudo-code for calculating subsequent OFC intervals, is presented in Table 4. The first column presents the algorithm for collecting increasing OFC intervals, while the second column presents the algorithm for collecting decreasing intervals.

Table 3 – Algorithm for calculating consecutive RHS intervals

collecting increasing RHS intervals	collecting decreasing RHS intervals
$k \coloneqq 0$	$k \coloneqq 0$
calculate β_j^+	calculate β_j^-
repeat	**repeat**
$b_j^k \coloneqq \begin{cases} b_j, & k=0 \\ I_e^{k-1}, & k \geq 1 \end{cases}$	$b_j^k \coloneqq \begin{cases} b_j, & k=0 \\ I_s^{k-1}, & k \geq 1 \end{cases}$
solve $LP(b_j \leftarrow b_j^k)$	solve $LP(b_j \leftarrow b_j^k)$
calculate $\xi_j^+(b_j^k)$	calculate $\xi_j^-(b_j^k)$
$I_s^k \coloneqq b_j^k$	$I_e^k \coloneqq b_j^k$
$I_e^k \coloneqq I_s^k + \xi_j^+(b_j^k)$	$I_s^k \coloneqq I_e^k - \xi_j^-(b_j^k)$
$I_{rate}^k \coloneqq SP^+$	$I_{rate}^k \coloneqq SP^-$
until ($I_e^k = \beta_j^+ \ or \ \xi_j^+(b_j^k) = \text{inf}$**)**	**until (**$I_e^k = \beta_j^- \ or \ \xi_j^-(b_j^k) = \text{inf}$**)**

In case of OFC parameters the objective value is changed exclusively by the change of the OFC parameter and the rate of change is equal to the value of variable i in the optimal solution (x_i) of the modified $LP(c_i \leftarrow c_i^k)$ problem.

The following steps are repeated while the maximum increase/decrease is finite:

- create and solve the modified $LP(c_i \leftarrow c_i^k)$ problem,
- calculate Type III range for the required direction,
- collect interval data.

Table 4 – Algorithm for calculating consecutive OFC intervals

collecting increasing OFC intervals	collecting decreasing OFC intervals
$k \coloneqq 0$	$k \coloneqq 0$
repeat	**repeat**
$c_i^k \coloneqq \begin{cases} c_i, & k=0 \\ I_e^{k-1}, & k \geq 1 \end{cases}$	$c_i^k \coloneqq \begin{cases} bc_i, & k=0 \\ I_s^{k-1}, & k \geq 1 \end{cases}$
solve $LP(c_i \leftarrow c_i^k)$	solve $LP(c_i \leftarrow c_i^k)$
calculate $\gamma_i^+(c_i^k)$	calculate $\gamma_i^-(c_i^k)$
$I_s^k \coloneqq c_i^k$	$I_s^k \coloneqq I_e^k - \gamma_i^-(c_i^k)$
$I_e^k \coloneqq I_s^k + \gamma_i^+(c_i^k)$	$I_e^k \coloneqq c_i^k$.
$I_{rate}^k \coloneqq x_i$	$I_{rate}^k \coloneqq x_i$
until ($\gamma_i^k = \text{inf}$**)**	**until (**$\gamma_i^k = \text{inf}$**)**

AIMMS implementation

From a practical point of view, creating a tool that can calculate and visualize the parametric objective value function requires a good solver and algorithmic and graphical user interface editing capabilities. Such a tool is provided by the AIMMS Prescriptive Analytics Platform, which is often used for solving commercial optimization problems in a wide range of industries including retail, consumer products, healthcare, oil and chemicals, steel production and agribusiness. (Roelofs, Bisschop, 2018)

AIMMS Prescriptive Analytics Platform is a tool for those with an Operations Research or Analytics background and offers a straightforward mathematical modelling environment and a wide range of available solvers. AIMMS also features an advanced graphical user interface editor which allows the creation of optimization application to individuals without a technical or analytics background.

For illustration purposes AIMMS version 4.42 was used to create the required mathematical models, implement the algorithms and create simple user interface, while CPLEX version 12.7.1 was used to solve the generated LP problems.

Four parametrized models were created in AIMMS:

- general parametrized linear program to solve both $LP(c_i \leftarrow c_i^k)$ and $LP(b_j \leftarrow b_j^k)$ problems,
- modified parametrized linear program to calculate maximal feasible increase/decrease of the RHS parameters,
- parametrized linear program to calculate Type III ranges of the right-hand-side parameters,
- parametrized linear program to calculate Type III ranges of the objective function coefficient parameters.

To collect all Type III intervals the implementation of the algorithms presented in Table 3 and 4 are needed. The algorithm collects the data related to the intervals, which then can be visualized using tables and line charts using pages created with the AIMMS user interface editor. The implemented solution consists of the following pages:

- LP definition page is the input page, where the desired LP can be formulated.
- LP solution page contains the original results provided by the solver.
- Two pages for presenting RHS interval data using table and line charts.
- Two pages for presenting OFC interval data using table and line charts.

Illustration example

In this section, a modified version of a simple LP case study taken from Anderson et al., (2015) will be used to illustrate the implementation of the suggested parametric analysis.

The problem consists in finding the optimal product mixt for a company producing three nut mixes for sale to grocery chains. The three mixes, referred to as the Regular Mix, the Deluxe Mix, and the Holiday Mix, are made by mixing different percentages of five types of nuts. For example, the Regular Mix consists of 15% almonds, 25% Brazil nuts, 25% filberts, 10% pecan and 20% walnuts. The company has an already purchased quantity from each raw material nut. Two make the LP related to the described problem dual degenerate, the amount of available almond and the margin of the Regular Mix were modified.

Table 5 contains the mix ratio of all the mixes. The last two column of Table 5 contains the margins related to the products calculated based on the production costs and prices related and the total of already received orders for each product. The last row of the table contains the available quantities of the raw material nuts.

Table 5 – Mix ratio, available quantities of the raw nuts and the margins of the nut mixes

	Almond	Brazil nut	Filbert	Pecan	Walnut	Margin (c)	Orders
Regular	15%	25%	25%	10%	25%	1,5	10000
Deluxe	20%	20%	20%	20%	20%	2	3000
Holiday	25%	15%	15%	25%	20%	2,25	5000
available quantity (b)	5390	7500	7500	6000	7500		

Let $x_R, x_D, x_H, c_R, c_D, c_H$ denote the produced quantities and the margins of the Regular, Deluxe and Holiday mixes. The optimal product mix can be calculated using the following LP:

$$\max(c_R x_R + c_R x_D + c_R x_H) \tag{3}$$
$$0{,}15x_R + 0{,}20x_D + 0{,}25x_H \leq b_A \tag{4}$$
$$0{,}25x_R + 0{,}20x_D + 0{,}15x_H \leq b_B \tag{5}$$
$$0{,}25x_R + 0{,}20x_D + 0{,}15x_H \leq b_F \tag{6}$$
$$0{,}10x_R + 0{,}20x_D + 0{,}25x_H \leq b_P \tag{7}$$
$$0{,}25x_R + 0{,}20x_D + 0{,}20x_H \leq b_W \tag{8}$$
$$x_R \geq 10000 \tag{9}$$
$$x_D \geq 3000 \tag{10}$$
$$x_H \geq 5000 \tag{11}$$

Table 6 contains the optimal product mix and the sensitivity intervals calculated by the solver.

Table 6 – Optimal solution and validity ranges related to the OFC parameters

	Optimal quantity	Original OFC	OFC lower limit	OFC upper limit
Regular	10000	1,5	-∞	1,5
Deluxe	13200	2	2	∞
Holiday	5000	2,25	-∞	2,5

Table 7 contains the shadow prices and the validity ranges related to the constraints.

Table 7 – Shadow prices and validity ranges for the available quantity constraints

	shadow price	lower limit	upper limit
Almond	10	3350	6500
Brazil	0	5890	∞
Filbert	0	5890	∞
Pecan	0	4890	∞
Walnut	0	6140	∞

Figure 1 contains the objective value function of the b_P parameter. The red dot marks the initial value of the RHS parameter, while the red line marks the validity range of the shadow price calculated by the solvers, and since the right bound is infinite only the left bound is indicated. The rate of change of the objective value function within the intervals are written on the lines. The chart resulted from the parametric analysis shows that if the amount of pecan was less than 2850 tons the problem would become infeasible. Between

2850 and 4210 tons the shadow price is 15 %/tons and for quantities higher than 4210 tons the shadow price is 0.

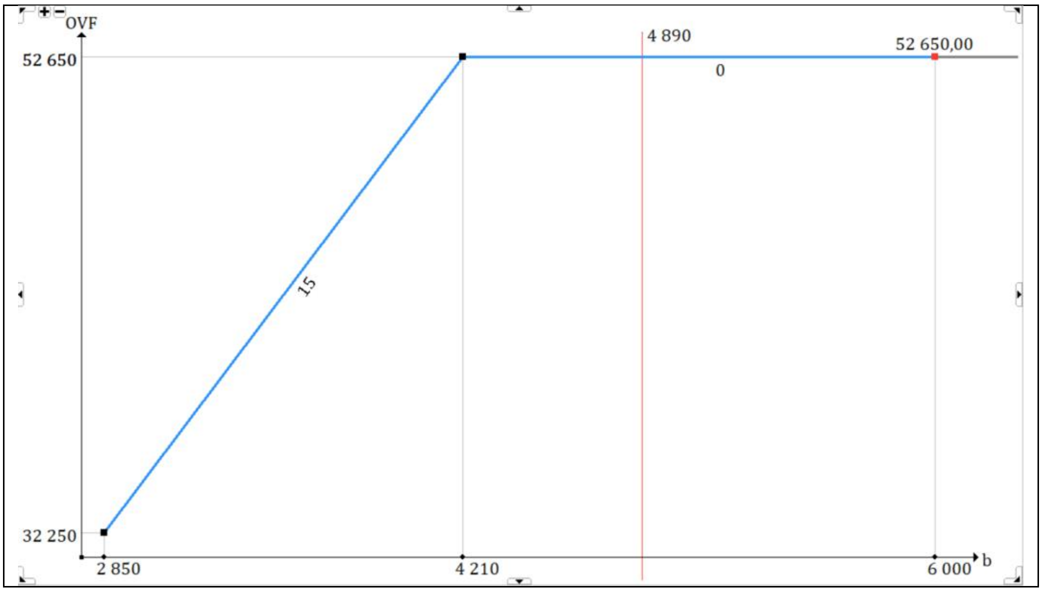

Figure 1 – Objective value function of the b_P parameter

Figure 2 shows the results of the OFC parametric analyses created by the AIMMS application. The table contains the interval start and end points, the rate of change of the objective value function within the interval, the value of the objective value function at the start and end point and the optimal solution related.

OFC sensitivity analysis results

							Optimal solution		
Variable	Intervals	Start	End	X	OVF Start	OVF End	Regular	Deluxe	Holiday
Regular	0	-inf	1,5	10000	-inf	52650	10000	13200	5000
	1	1,5	inf	23600	52650	inf	23600	3000	5000
Deluxe	0	-inf	2	3000	-inf	52650	23600	3000	5000
	1	2	inf	13200	52650	inf	10000	13200	5000
Holiday	0	-inf	2,25	5000	-inf	52650	10000	13200	5000
	1	2,25	2,5	5000	52650	53900	10000	13200	5000
	2	2,5	inf	13160	53900	inf	10000	3000	13160

Figure 2 – part of the AIMMS page with table containing all the OFC intervals

Compared with the results provided by the basic AIMMS solver it can be noticed that the solver calculated misleading information about the validity intervals of the c_R and c_D OFC parameters. The misleading information is the consequence of degeneracy. The presented method provides correct LP sensitivity information in case of degeneracy as well. Furthermore, the implemented AIMMS algorithm presents the objective value function for the whole feasible range of any required OFC and RHS parameter.

Conclusion

In this paper, the implementation of parametric analysis of LP models in AIMMS to support operations management decision making is presented. The information provided by the suggested method describes the objective value function in the feasible range of any OFC and RHS parameter.

The benefits of this information are twofold:

- Since all values of the objective value function are known, there are no misleading results as a consequence of degeneracy. If appropriate, the left and right shadow prices (slope of the objective value function in case of decrease and increase) are given, and the correct ranges of all parameters are calculated.
- The traditional sensitivity ranges provided by most commercial LP solvers are completed with further information. In our case, the effect of the parameter change is known, not only in the close neighbourhood of the original value, but also in the whole feasible region.

The presented method can be used to support OM decision whenever the problem of the allocation of scarce resources must be solved, and LP models can properly describe or approximate the problem. The created AIMMS application, beside showing the consecutive Type III intervals in a table format, also contains a simple graphical presentation of the results using line charts, to create a better overview of the decision situation. The presented objective value function of any of the critical parameters can help operation managers to see directly the effect of planned or random parameter changes, or the possible consequences of the inaccuracy of data applied in the operation planning phase.

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